Understanding Toussaint's Maximally Even Rhythms

Background

Godfried Toussaint, a prominent computer scientist and musicologist, has made significant contributions to the understanding and application of Euclidean rhythms in music. His exploration into maximally even rhythms has revealed fascinating patterns that occur naturally in various musical traditions around the world.

Toussaint was born in Belgium and spent his formative years in Bolivia, Colombia, and Brazil. He began his musical journey with classical guitar before discovering African percussion under the tutelage of David Thiau in Montreal in 1978. Toussaint furthered his studies in African percussion with Babatunde Olatunji in Vancouver and explored Latin percussion with Pere Gomez in Barcelona.

In 2013, Godfried Toussaint published a book titled "The Geometry of Musical Rhythm," subtitled "What Makes a 'Good' Rhythm Good?" In this book, he analyzed the symmetry of world rhythms such as the African Shiko, Soukous, and Gahu, as well as the Afro-Cuban son clave, rumba clave, and Brazilian bossa nova clave. (These are presented in detail under the “5 Notes Per Bar” section.) He presented these rhythms with each 16th note as points around a circle, representing the rhythms as polygons whose sides connect the 'onsets' (referred to as 'attacks' in my charts, meaning the same thing in this context, i.e., a note event as opposed to a rest event).

At least one rhythm from each group of examples that follow are played in the follow video.

https://youtu.be/7OrDmWtwfwE

In the book, Toussaint says that the first notion of a "good" rhythm is a maximally even rhythm. Since this wasn’t the only condition of a “good” rhythm, he wasn’t restrictive about achieving the absolute maximum amount of space between notes. He came up with rhythms that were, for example, still relatively evenly spaced, but not necessarily where every note was as far away from others as possible. Here are two examples where the rhythms are relatively even but not necessarily maximally even, as the first one is 3 + 4 + 4 + 3 + 2 and second is 1 + 2 + 4 + 2 + 5.

Sample Toussaint less-than-maximal rhythms

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1		X			X				X				X			X
2	X				X		X				X		X			

Toussant also did not limit his rhythms to those that were in 16th note subdivisions. For example, here are some examples of even rhythms which are based on other numbers of beats per bar:

Nyunga-Nyunga Mbira (6-beat cycle)

1	2	3	4	5	6
X		X	X	X	

Cinquillo (8 beat-cycle)

1	&	2	&	3	&	4	&
X		X	X		X	X	

Fume-Fume (12-beat cycle)

1	&	a	2	&	a	3	&	a	4	&	a
X		X		X			X		X		

Bembe (12-beat cycle)

1	&	a	2	&	a	3	&	a	4	&	a
X		X		X	X		X		X		X

For this article, we will focus only on rhythms that are derived from having the notes be as maximally spaced as possible within the bar, including how each bar wraps around to the next, and we will use 16th notes as our beat cycle.

For example, for 5 notes in a bar, 5/16 = 0.3125. Each note will be a little more than 3 sixteenth notes to be maximally spaced, but since 16th notes is our basic subdivision, we can only make 3 of them 3-sixteenths long and 1 of them 4-sixteenths long to make a total of 16. We refer to this as 4 + 3 + 3 + 3, and they can be in any order, such as 3 + 3 + 4 + 3, etc.

We will then rotate each combination through each possible starting point within the bar, as in starting on the 1, then on the “a” of 1, etc. We will not use standard notation, but a graphic representation of each rhythm so that we can a) include readers who are not familiar with standard notation and b) better conceive of the space between notes for each rhythm presented.

Using “Vivir Mi Vida” as an Example

I first became interested in this topic when I was asked to play Marc Anthony's "Vivir Mi Vida" on a gig. The introduction has the piano playing rhythm #16 in the "5 attacks per bar" chart shown below in the "5 Notes Per Bar" section. When I heard this rhythm I asked myself "Why does this sound so good?" and I think it's because:

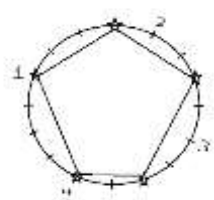
1. The notes are maximally spaced
2. 3 out of 5 of the notes are anticipated or delayed
3. The whole rhythm ingeniously supports the 3-2 clave on which the song is based.

So you can see what I mean, here is the rhythm:

Rhythm of "Vivir Mi Vida" vs. 3-2 Son Clave

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
#16				X			X			X			X			X
3-2 son clave	X			X			X				X		X			

The first, second, and fourth of the 5 notes of rhythm #16 coincide exactly with 3 of the 5 notes of the 3-2 son clave, which is a rhythm Toussaint loved, by the way. It's not "maximally even" because the last two notes are closer together, but it is even and symmetrical, which can be seen if we plot this rhythm as a polygon, similar to how Toussaint did:



Yet another cool feature of the introduction to "Vivir Mi Vida" is that the very first beat is played as well, which makes a 4:3 polyrhythm for the first full bar, until the final note extends into the next bar for 4 sixteenths, like it did on rhythm 16 above.

Introduction of "Vivir Mi Vida"

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
	X			X			X			X			X			X

If each note continued as a 3 + 3 + 3 etc. sequence, it would have taken 3 bars to resolve, as follows:

4:3 polyrhythm 3-bar sequence

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
	X			X			X			X			X			X

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
			X			X			X			X			X	

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
		X			X			X			X			X		

Even though the tune doesn’t continue this way, as a listener we don’t feel that the 3 + 3 + 3 + 4 pattern used is a disadvantage in any way. In fact, it is very satisfying because it:

1. Fits the most common time signature, 4/4
2. Is felt as even because the difference that the extra sixteenth in the “4” of the 3 + 3 + 3 + 4 is hardly noticeable to the casual listener
3. Allows for each bar to be anticipated by a 16th which gives forward momentum and upliftment to the listener

Using “Vivir Mi Vida” as an example really illustrates how satisfying all of the following maximally even rhythms are. They are all in 4/4, maximally spaced, and provide a balance of notes that are ‘on’ the beat and ‘off’ the beat. You may even recognize some of them as the drum beats, guitar riffs, or vocal melodies of famous songs. They are musically satisfying because subconsciously we feel their mathematical elegance, thanks to Toussaint.

Playing through all of these rhythms is a great exercise for musicians, but in the video I only gave 3 examples per combination.

Some may give you compositional inspiration. I hope all of them bring you enjoyment. Let’s dive into each number of notes per bar from 3 to 9.

3 Notes Per Bar

For 3 maximally spaced notes in a bar, we must use some combination of 6 + 5 + 5. There are three possible combinations here, of which the other two are 5 + 6 + 5 and 6 + 5 + 5. Representing 6 + 5 + 5 where 1 is a note and 0 is a rest, we get 1000001000010000. Since the sequence is going to rotate where it begins within the bar, as in the next one being 0100000100001000, we will hit upon the other two combinations automatically. Therefore the total number of possibilities of 3 maximally spaced notes within a bar is **16**.

3 attacks in a bar

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X					X					X					
2		X					X					X				
3			X					X					X			
4				X					X					X		
5					X					X					X	
6						X					X					X
7	X						X					X				
8		X						X					X			
9			X						X					X		
10				X						X					X	
11					X						X					X
12	X					X						X				
13		X					X						X			
14			X					X						X		
15				X					X						X	
16					X					X						X

4 Notes Per Bar

For 4 maximally spaced notes in a bar, we must use 4 + 4 + 4 + 4. These can start on any of the 4 sixteenth-note subdivisions of a beat. Thus the total number of possibilities of 3 maximally spaced notes within a bar is **4**.

4 attacks in a bar

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X				X				X				X			
2		X				X				X				X		
3			X				X				X				X	
4				X				X				X				X

5 Notes Per Bar

For 5 maximally spaced notes in a bar, we must use some combination of 4 + 3 + 3 + 3. Similar to how there is only one "6" in the above example, the fact there is only one "4" in this example means that we will automatically hit on the other possible combinations which are a) 3 + 4 + 3 + 3 b) 3 + 3 + 4 + 3 and c) 3 + 3 + 3 + 4. Thus the total number of possibilities of 5 maximally spaced notes is **16**.

5 attacks in a bar

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X				X			X			X			X		
2		X				X			X			X			X	
3			X				X			X			X			X
4	X			X				X			X			X		
5		X			X				X			X			X	
6			X			X				X			X			X
7	X			X			X				X			X		
8		X			X			X				X			X	
9			X			X			X				X			X
10	X			X			X			X				X		
11		X			X			X			X				X	
12			X			X			X			X				X
13	X			X			X			X			X			
14		X			X			X			X			X		
15			X			X			X			X			X	
16				X			X			X			X			X

Godfried Toussaint placed significant focus on rhythms with 5 notes per bar in his research on Euclidean rhythms. The structure of distributing 5 notes evenly within a 16-beat cycle presents a clear and compelling case for demonstrating the mathematical principles of Euclidean rhythms.

In addition, Toussaint's research highlighted the cultural significance of these rhythms, showing how they are not just mathematical curiosities but deeply embedded in the musical traditions of many cultures. Rhythms with 5 beats (often represented as Euclidean rhythms) are prevalent in various musical traditions around the world, including African, Latin American, and Balkan music traditions. Here are what he calls the 6 "fundamental African and Latin American rhythms." Every single one of them has

5 attacks per bar, and the last one, the bossa nova rhythm is exactly the same as rhythm #7 as presented below under “5 Attacks Per Bar.”

6 Fundamental African & Latin American Rhythms

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
Shiko	X				X		X				X		X			
Son	X			X			X				X		X			
Rumba	X			X				X			X		X			
Soukous	X			X			X				X	X				
Gahu	X			X			X				X				X	
Bossa-nova	X			X			X				X			X		

6 Notes Per Bar

For 6 maximally spaced notes in a bar, we must use some combination of 2 + 2 + 3 + 3 + 3 + 3. There are 3 possibilities:

1) twos are separated by 0 or 4 sets of three

2) twos are separated by 1 or 3 sets of three

3) two are separated by 2 sets of three

Since there are 3 unique combinations and 16 ways to shift each one within 16 beats, the total number of possible combinations of 6 maximally spaced notes is 3 x 16 = **48.**

6 attacks in a bar, group 1: twos separated by either 0 or 4 sets of three

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X		X		X			X			X			X		
2		X		X		X			X			X			X	
3			X		X		X			X			X			X
4	X			X		X		X			X			X		
5		X			X		X		X			X			X	
6			X			X		X		X			X			X
7	X			X			X		X		X			X		
8		X			X			X		X		X			X	
9			X			X			X		X		X			X
10	X			X			X			X		X		X		
11		X			X			X			X		X		X	
12			X			X			X			X		X		X
13	X			X			X			X			X		X	
14		X			X			X			X			X		X
15	X		X			X			X			X			X	
16		X		X			X			X			X			X

6 attacks in a bar, group 2: twos separated by either 1 or 3 sets of three

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X		X			X		X			X			X		
2		X		X			X		X			X			X	
3			X		X			X		X			X			X
4	X			X		X			X		X			X		
5		X			X		X			X		X			X	
6			X			X		X			X		X			X
7	X			X			X		X			X		X		
8		X			X			X		X			X		X	
9			X			X			X		X			X		X
10	X			X			X			X		X			X	
11		X			X			X			X		X			X
12	X		X			X			X			X		X		
13		X		X			X			X			X		X	
14			X		X			X			X			X		X
15	X			X		X			X			X			X	
16		X			X		X			X			X			X

6 attacks in a bar, group 3: twos separated by 2 sets of three

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X		X			X			X		X			X		
2		X		X			X			X		X			X	
3			X		X			X			X		X			X
4	X			X		X			X			X		X		
5		X			X		X			X			X		X	
6			X			X		X			X			X		X
7	X			X			X		X			X			X	
8		X			X			X		X			X			X
9	X		X			X			X		X			X		
10		X		X			X			X		X			X	
11			X		X			X			X		X			X
12	X			X		X			X			X		X		
13		X			X		X			X			X		X	
14			X			X		X			X			X		X
15	X			X			X		X			X			X	
16		X			X			X		X			X			X

7 Notes Per Bar

For 7 maximally spaced notes in a bar, we must use some combination of 3 + 3 + 2 + 2 + 2 + 2 + 2. There are only unique combinations which are

1) threes are separated by any either 0 or 5 sets of two

2) when the threes are either separated by 1 or 4 sets of two

3) when the threes are separated by either 2 or 3 sets of two

Since there are 3 unique combinations and 16 ways to shift each one within 16 beats, the total number of possible combinations of 7 maximally spaced notes is 3 x 16 = **48**.

7 attacks in a bar, group 1: threes are separated by either 0 or 5 sets of two

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X			X			X		X		X		X		X	
2		X			X			X		X		X		X		X
3	X		X			X			X		X		X		X	
4		X		X			X			X		X		X		X
5	X		X		X			X			X		X		X	
6		X		X		X			X			X		X		X
7	X		X		X		X			X			X		X	
8		X		X		X		X			X			X		X
9	X		X		X		X		X			X			X	
10		X		X		X		X		X			X			X
11	X		X		X		X		X		X			X		
12		X		X		X		X		X		X			X	
13			X		X		X		X		X		X			X
14	X			X		X		X		X		X		X		
15		X			X		X		X		X		X		X	
16			X			X		X		X		X		X		X

7 attacks in a bar, group 2: threes are separated by either 1 or 4 sets of two

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X			X		X			X		X		X		X	
2		X			X		X			X		X		X		X
3	X		X			X		X			X		X		X	
4		X		X			X		X			X		X		X
5	X		X		X			X		X			X		X	
6		X		X		X			X		X			X		X
7	X		X		X		X			X		X			X	
8		X		X		X		X			X		X			X
9	X		X		X		X		X			X		X		
10		X		X		X		X		X			X		X	
11			X		X		X		X		X			X		X
12	X			X		X		X		X		X			X	
13		X			X		X		X		X		X			X
14	X		X			X		X		X		X		X		
15		X		X			X		X		X		X		X	
16			X		X			X		X		X		X		X

7 attacks in a bar, group 3: threes are separated by either 2 or 3 sets of two

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X			X		X		X			X		X		X	
2		X			X		X		X			X		X		X
3	X		X			X		X		X			X		X	
4		X		X			X		X		X			X		X
5	X		X		X			X		X		X			X	
6		X		X		X			X		X		X			X
7	X		X		X		X			X		X		X		
8		X		X		X		X			X		X		X	
9			X		X		X		X			X		X		X
10	X			X		X		X		X			X		X	
11		X			X		X		X		X			X		X
12	X		X			X		X		X		X			X	
13		X		X			X		X		X		X			X
14	X		X		X			X		X		X		X		
15		X		X		X			X		X		X		X	
16			X		X		X			X		X		X		X

8 Notes Per Bar

For 8 maximally spaced notes in a bar, we must use some combination of 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2. This can be presented as 1 & 2 & 3 & 4 & or the “e” and “a” of each beat. Therefore the total number of possibilities of 3 maximally spaced notes within a bar is **2**.

8 attacks in a bar

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X		X		X		X		X		X		X		X	
2		X		X		X		X		X		X		X		X

9 Notes Per Bar

For 9 maximally spaced notes in a bar, we must use some combination of 1 + 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2. There are four possibilities:

1) ones separated by either 0 or 7 sets of twos

2) ones separated by either 1 or 6 sets of twos

3) ones separated by either 2 or 5 sets of twos

4) ones separated by either 3 or 4 sets of twos

Because each set of possibilities can be rotated through 16 different starting points within the bar, the total number of possible combinations of 9 maximally spaced notes is 4 x 16 = **64**.

9 attacks in a bar, group 1: ones are separated by either 0 or 7 sets of two

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X	X	X		X		X		X		X		X		X	
2	X		X	X	X		X		X		X		X		X	
3	X		X		X	X	X		X		X		X		X	
4	X		X		X		X	X	X		X		X		X	
5	X		X		X		X		X	X	X		X		X	
6	X		X		X		X		X		X	X	X		X	
7	X		X		X		X		X		X		X	X	X	
8	X		X		X		X		X		X		X		X	X
9	X	X		X		X		X		X		X		X		X
10		X	X	X		X		X		X		X		X		X
11		X		X	X	X		X		X		X		X		X
12		X		X		X	X	X		X		X		X		X
13		X		X		X		X	X	X		X		X		X
14		X		X		X		X		X	X	X		X		X
15		X		X		X		X		X		X	X	X		X
16		X		X		X		X		X		X		X	X	X

9 attacks in a bar, group 2: ones are separated by either 1 or 6 sets of two

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X	X		X	X		X		X		X		X		X	
2		X	X		X	X		X		X		X		X		X
3	X		X	X		X	X		X		X		X		X	
4		X		X	X		X	X		X		X		X		X
5	X		X		X	X		X	X		X		X		X	
6		X		X		X	X		X	X		X		X		X
7	X		X		X		X	X		X	X		X		X	
8		X		X		X		X	X		X	X		X		X
9	X		X		X		X		X	X		X	X		X	
10		X		X		X		X		X	X		X	X		X
11	X		X		X		X		X		X	X		X	X	
12		X		X		X		X		X		X	X		X	X
13	X		X		X		X		X		X		X	X		X
14	X	X		X		X		X		X		X		X	X	
15		X	X		X		X		X		X		X		X	X
16	X		X	X		X		X		X		X		X		X

9 attacks in a bar, group 3: ones are separated by either 2 or 5 sets of two

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X	X		X		X	X		X		X		X		X	
2		X	X		X		X	X		X		X		X		X
3	X		X	X		X		X	X		X		X		X	
4		X		X	X		X		X	X		X		X		X
5	X		X		X	X		X		X	X		X		X	
6		X		X		X	X		X		X	X		X		X
7	X		X		X		X	X		X		X	X		X	
8		X		X		X		X	X		X		X	X		X
9	X		X		X		X		X	X		X		X	X	
10		X		X		X		X		X	X		X		X	X
11	X		X		X		X		X		X	X		X		X
12	X	X		X		X		X		X		X	X		X	
13		X	X		X		X		X		X		X	X		X
14	X		X	X		X		X		X		X		X	X	
15		X		X	X		X		X		X		X		X	X
16	X		X		X	X		X		X		X		X		X

9 attacks in a bar, group 4: ones are separated by either 3 or 4 sets of two

	1	e	&	a	2	e	&	a	3	e	&	a	4	e	&	a
1	X	X		X		X		X	X		X		X		X	
2		X	X		X		X		X	X		X		X		X
3	X		X	X		X		X		X	X		X		X	
4		X		X	X		X		X		X	X		X		X
5	X		X		X	X		X		X		X	X		X	
6		X		X		X	X		X		X		X	X		X
7	X		X		X		X	X		X		X		X	X	
8		X		X		X		X	X		X		X		X	X
9	X		X		X		X		X	X		X		X		X
10	X	X		X		X		X		X	X		X		X	
11		X	X		X		X		X		X	X		X		X
12	X		X	X		X		X		X		X	X		X	
13		X		X	X		X		X		X		X	X		X
14	X		X		X	X		X		X		X		X	X	
15		X		X		X	X		X		X		X		X	X
16	X		X		X		X	X		X		X		X		X

Applications and Broader Implications

Toussaint's research on maximally even rhythms extends beyond academic interest. These rhythmic patterns have practical applications in contemporary music composition, algorithmic music generation, and even in understanding traditional music forms.

In traditional African and Afro-Cuban music, Euclidean rhythms help explain the natural balance found in many percussion patterns. For instance, the bell patterns in African drumming often align with maximally even distributions, creating rhythms that are both complex and intuitively understandable.

In modern music, composers and producers use Euclidean rhythms to generate intricate beats and loops. The algorithmic nature of these rhythms allows for the creation of patterns that feel natural and groovy, avoiding the mechanical feel often associated with purely computer-generated music.

Conclusion

Godfried Toussaint’s research on maximally even rhythms provides a fascinating insight into the mathematical and cultural foundations of rhythmic patterns. By understanding and applying these rhythms, musicians can create patterns that are both mathematically elegant and musically satisfying. Toussaint's work highlights the universality of these rhythms, demonstrating their prevalence across various musical traditions and their potential for innovation in contemporary music.

P.S. If you are interested in 10, 11, 12, 13, 14, or 15 maximally spaced notes per bar, there are too many combinations to be presented graphically as we have done above. Instead we used the number 1 to represent a note (or "attack" as we've been calling it) and 0 to represent a rest (or simply a beat on which a note that was previously struck would continue on) as in 1111-1010-1010-1010 for a sample 10-note per bar sequence. Here is the pdf link for it.

P.P.S. If you want to know what the combinations of 3 to 11 look like in a swing or shuffle groove (i.e. with 8th-note triplets, or 12-subdivisions per bar), click here.